

GENERAL POSITION PROPERTIES IN FIBERWISE GEOMETRIC TOPOLOGY

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INTRODUCTION

The classical Lefschetz-Menger-Nöbeling-Pontrjagin-Tolstova Embedding Theorem asserts that each n -dimensional compact metric space embeds into the $(2n + 1)$ -dimensional Euclidean space \mathbb{R}^{2n+1} . A parametric version of this result was proved by B.Pasynkov [Pa1]: any n -dimensional map $p : K \rightarrow M$ between metrizable compacta with $\dim M = m$ embeds into the projection $\text{pr}_M : M \times \mathbb{R}^{2n+1+m} \rightarrow M$ in the sense that there is an embedding $e : K \rightarrow M \times \mathbb{R}^{2n+1+m}$ with $\text{pr}_M \circ e = p$. We recall that a map $p : X \rightarrow Y$ is n -dimensional if $\dim p^{-1}(y) \leq n$ for all $y \in Y$ (0-dimensional maps are called sometimes *light* maps).

The key ingredient of Pasynkov's proof consists in constructing a map $f : K \rightarrow \mathbb{R}^{2n+1+m}$ that is injective on each fiber $p^{-1}(y)$, $y \in M$, of p . Actually, Pasynkov proved that the set of all such maps is dense and G_δ in the function space $C(K, \mathbb{R}^{2n+1+m})$.

In this paper we introduce the m -DD n -property and show that the Pasynkov result remains true if \mathbb{R}^{2n+1+m} is replaced by any completely metrizable LC^{m+n} -space X possessing this property. Let us note that the m -DD n -property is a parametric version of the classical disjoint n -disks property DD n P which plays a crucial role in characterizing finite and infinite-dimensional manifolds, see [We]. We shall also give convenient "arithmetic" tools for establishing the m -DD n -property of products and obtain on this base simple proofs of some classical and recent results on (fiber) embeddings. In particular, the Pasynkov theorem mentioned above, as well as the results of P. Bowers [Bow] and Y. Sternfeld [Ste] on embedding into product of dendrites follow from our general approach. Moreover, the arithmetics of the m - $\overline{\text{DD}}^{\{n,k\}}$ -properties established in our paper generalizes some results of W. Mitchell [WM], R. Daverman [Da1] and D. Halverson [Hal], [DH].

1. SURVEY OF PRINCIPAL RESULTS

Throughout the paper m, n, k will stand for non-negative integers or ∞ . We extend the arithmetic operations from $\omega = \{0, 1, 2, \dots\}$ onto $\omega \cup \{\infty\}$ letting $\infty = \infty + \infty = \infty + n = n + \infty = \infty - n$ for any $n \in \omega$. \mathbb{I} denotes the unit interval $[0, 1]$ and \mathbb{Q} the set of rational numbers on the real line \mathbb{R} . By a simplicial complex we shall always mean the geometric realization of an abstract simplicial complex equipped with the CW -topology. All topological spaces are assumed to be *Tychonoff* and all maps continuous.

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By an ANR-space we mean a metrizable space X which is a neighborhood retract of every metrizable space M containing X as a closed subspace. It is well-known (see [Bors] or [Hu]) that a metrizable space X is an ANR if and only if it is an $\text{ANE}[\infty]$ for the class of metrizable spaces. We recall that a space X is called an $\text{ANE}[n]$ for a class \mathcal{C} of spaces if every map $f : A \rightarrow X$ defined on a closed subset of a space $C \in \mathcal{C}$ with $\dim C \leq n$ can be extended to a continuous map $\bar{f} : U \rightarrow X$ defined on some neighborhood U of A in X .

Following [EW], we define a subset A of a space X to be *relative* LC^n in X if given $x \in X$, $k < n + 2$, and a neighborhood U of x there is a neighborhood $V \subset U$ of x such that each map $f : \partial \mathbb{I}^k \rightarrow A \cap V$ extends to a map $\bar{f} : \mathbb{I}^k \rightarrow U \cap A$. A space X is an LC^n -space if it is relative LC^n in X . According to [Hu, V.2.1], a metrizable space X is LC^n for a finite number n if and only if X is $\text{ANE}[n + 1]$ for the class of metrizable spaces.

1.1. m - DD^n -property and fiber embeddings. We recall that a space X has DD^nP , the *disjoint n -disks property*, if any two maps $f, g : \mathbb{I}^n \rightarrow X$ from the n -dimensional cube $\mathbb{I}^n = [0, 1]^n$ can be approximated by maps with disjoint images. A parametric version of this property says that the same can be done for a continuous family $f_z, g_z : \mathbb{I}^n \rightarrow X$ of maps parameterized by points z of some space M . More precisely, given a compact space M , we shall say that a space X has the *M -parametric disjoint n -disk property* (briefly, the *M - DD^n -property*) if any two maps $f, g : M \times \mathbb{I}^n \rightarrow X$ can be uniformly approximated by maps $f', g' : M \times \mathbb{I}^n \rightarrow X$ such that for any $z \in M$ the images $f'(\{z\} \times \mathbb{I}^n)$ and $g'(\{z\} \times \mathbb{I}^n)$ are disjoint.

We are mostly interested in the particular case of this property with $M = \mathbb{I}^m$ being the m -dimensional cube. In this case we write m - DD^n instead of \mathbb{I}^m - DD^n . In the extremal cases when m or n is zero, the m - DD^n -property turns out to be very familiar. Namely, the 0 - DD^n -property is nothing else but the classical disjoint n -disks property, while the m - DD^0 -property is well-known to specialists in fixed point and coincidence theories: a space X has the m - DD^0 -property iff any two maps $f, g : \mathbb{I}^m \rightarrow X$ can be approximated by maps with disjoint graphs!

It is well known (see [To2] or [Ed]) that all one-to-one maps from a metrizable n -dimensional compactum K into a completely metrizable LC^{n-1} -space X possessing the DD^nP -property form a dense G_δ -set in the function space $C(K, X)$. Our first principal result is just a parametric version of this embedding theorem.

Theorem 1. *A completely metrizable LC^{m+n} -space X has the m - DD^n -property if and only if for every perfect map $p : K \rightarrow M$ between finite-dimensional metrizable spaces with $\dim M \leq m$ and $\dim(p) \leq n$ the function space $C(K, X)$ contains a dense G_δ -set of maps $f : K \rightarrow X$ that are injective on each fiber $p^{-1}(z)$, $z \in M$.*

The function space $C(K, X)$ appearing in this theorem is endowed with the source limitation topology whose neighborhood base at a given function $f \in C(K, X)$ consists of the sets

$$B_\rho(f, \varepsilon) = \{g \in C(K, X) : \rho(g, f) < \varepsilon\},$$

where ρ runs over all continuous pseudometrics on X and $\varepsilon : K \rightarrow (0, \infty)$ runs over continuous positive functions on K . Here, the symbol $\rho(f, g) < \varepsilon$ means that $\rho(f(x), g(x)) < \varepsilon(x)$ for all $x \in K$. To the best of our knowledge, the notion of source limitation topology was introduced in the literature (see for example, [Kr], [Mu], [Mc]) only for metrizable spaces X . In such a case, for a fixed compatible

metric ρ on X , the sets $B_\rho(f, \varepsilon)$, $\varepsilon \in C(K, (0, \infty))$ and $f \in C(K, X)$, form a base for a topology \mathcal{T}_ρ on $C(K, X)$. If K is paracompact, then the topology \mathcal{T}_ρ does not depend on the metric ρ [Kr]. Moreover, \mathcal{T}_ρ has the Baire property provided K is paracompact and X is completely metrizable [Mu]. According to Lemma ?? below, \mathcal{T}_ρ coincides with the topology obtained from our definition provided K is paracompact and X metrizable. Therefore, the source limitation topology on $C(K, X)$ also has the Baire property when K is paracompact and X is completely metrizable. In the sequel, we will use our more general definition (in terms of pseudometrics) and, unless stated otherwise, all function spaces will be considered with this topology.

In fact, finite-dimensionality of the spaces K, M in Theorem 1 can be replaced by the C -space property. We recall that a topological space X is defined to be a C -space if for any sequence $\{\mathcal{V}_n : n \in \omega\}$ of open covers of X there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of disjoint families of open sets in X such that each \mathcal{U}_n refines \mathcal{V}_n and $\bigcup \{\mathcal{U}_n : n \in \omega\}$ is a cover of X . It is known that every finite-dimensional paracompact space (as well as every hereditarily paracompact countable-dimensional space) is a C -space and normal C -spaces are weakly infinite-dimensional, see [En1, §6.3].

Theorem 2. *A completely metrizable locally contractible space X has the m -DD n -property if and only if for every perfect map $p : K \rightarrow M$ between metrizable C -spaces with $\dim M \leq m$ and $\dim(p) \leq n$ the function space $C(K, X)$ contains a dense G_δ -set of maps $f : K \rightarrow X$ that are injective on each fiber $p^{-1}(z)$, $z \in M$.*

1.2. Δ -dimension of maps. There is a natural temptation to remove the dimensional restrictions on the spaces K, M from Theorems 1 and 2. This indeed can be done if we replace the usual dimension $\dim(p)$ of the map p with the so-called Δ -dimension $\dim_\Delta(p)$ (coinciding with $\dim(p)$ for perfect maps p between finite-dimensional metrizable spaces.)

By definition, the Δ -dimension $\dim_\Delta(p)$ of a map $p : X \rightarrow Y$ between Tychonoff spaces is equal to the smallest cardinal number τ for which there is a map $g : X \rightarrow \mathbb{I}^\tau$ such that the diagonal product $f\Delta g : X \rightarrow Y \times \mathbb{I}^\tau$ is a light map. The Δ -dimension $\dim_\Delta(p)$ is a well-defined cardinal function non-exceeding the weight $w(X)$ of X (because we always can take g to be an embedding in the Tychonoff cube $\mathbb{I}^{w(X)}$).

The following important result describing the interplay between the dimension and Δ -dimension of perfect maps is actually a reformulation of results due to B. Pasynkov [Pa2], M. Tuncali, V. Valov [TV], and M. Levin [Lev], see Section ??.

Proposition 1. *Let $f : X \rightarrow Y$ be a perfect map between paracompact spaces. Then*

- (1) $\dim(f) \leq \dim_\Delta(f)$;
- (2) $\dim_\Delta(f) = 0$ if and only if f is a light map;
- (3) $\dim_\Delta(f) \leq \omega$ if X is submetrizable;
- (4) $\dim_\Delta(f) = \dim(f)$ if X is submetrizable and Y is a C -space;
- (5) $\dim_\Delta(f) \leq \dim(f) + 1$ if the spaces X, Y are compact and metrizable.

We recall that a topological space X is *submetrizable* if it admits a continuous metric (equivalently, admits a bijective continuous map onto a metrizable space). The following theorem is a version of Theorem 1 with $\dim(p)$ replaced by $\dim_\Delta(p)$.

Theorem 3. *A completely metrizable ANR-space X has the m -DD n -property if and only if for every perfect map $p : K \rightarrow M$ between submetrizable paracompact*

spaces with $\dim M \leq m$ and $\dim_{\Delta}(p) \leq n$ the function space $C(K, X)$ contains a dense G_{δ} -set of maps $f : K \rightarrow X$ that are injective on each fiber $p^{-1}(z)$, $z \in M$.

1.3. The $m\text{-}\overline{\text{DD}}^n$ -property and a general fiber embedding theorem. In fact, it is more convenient to work not with the $m\text{-DD}^n$ -property, but with its homotopical version defined as follows:

Definition 1. A space X has the $m\text{-}\overline{\text{DD}}^n$ -property if for any open cover \mathcal{U} of X and any two maps $f, g : \mathbb{I}^m \times \mathbb{I}^n \rightarrow X$ there are maps $f', g' : \mathbb{I}^m \times \mathbb{I}^n \rightarrow X$ such that

- f' is \mathcal{U} -homotopic to f ;
- g' is \mathcal{U} -homotopic to g ;
- $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^n) = \emptyset$ for all $z \in \mathbb{I}^m$.

We recall that two maps $f, g : K \rightarrow X$ are said to be \mathcal{U} -homotopic (briefly, $f \sim_{\mathcal{U}} g$), where \mathcal{U} is a cover of X , if there is a homotopy $h : K \times [0, 1] \rightarrow X$ such that for every $x \in K$ we have $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ and $h(\{x\} \times [0, 1])$ is contained in some $U \in \mathcal{U}$. It is clear that any \mathcal{U} -homotopic maps $f, g : K \rightarrow X$ are \mathcal{U} -near (i.e., for each point $z \in K$ the set $\{f(z), g(z)\}$ lies in some $U \in \mathcal{U}$).

The notion of a \mathcal{U} -homotopy has a pseudometric counterpart. Given a continuous pseudometric ρ on X and a continuous map $\varepsilon : K \rightarrow (0, \infty)$ we shall say that two maps $f, g : K \rightarrow X$ are ε -homotopic if there is a homotopy $h : K \times [0, 1] \rightarrow X$ such that $h(z, 0) = f(z)$, $h(z, 1) = g(z)$ and $\text{diam}_{\rho}(h(\{z\} \times [0, 1])) < \varepsilon(z)$ for all $z \in K$. In this case h is called an ε -homotopy.

The relation between the $m\text{-DD}^n$ -property and its homotopical version is described by next proposition.

Proposition 2. *Each space X with the $m\text{-}\overline{\text{DD}}^n$ -property has the $m\text{-DD}^n$ -property. Conversely, each LC^{n+m} -space X possessing the $m\text{-DD}^n$ -property has the $m\text{-}\overline{\text{DD}}^n$ -property.*

Proposition 2 follows from the well-known property of LC^n -spaces which asserts that for any open cover \mathcal{U} of an LC^n -space X with $n < \infty$ there is another open cover \mathcal{V} of X such that two maps $f, g : \mathbb{I}^n \rightarrow X$ are \mathcal{U} -homotopic provided they are \mathcal{V} -near, see Lemma ??.

Thus, in the realm of LC^{m+n} -spaces both the $m\text{-}\overline{\text{DD}}^n$ -property and the $m\text{-DD}^n$ -property are equivalent. The advantage of the $m\text{-}\overline{\text{DD}}^n$ -property is that it works for spaces without a nice local structure, while the $m\text{-DD}^n$ -property is applicable only for LC^k -spaces with sufficiently large k . In particular, using the $m\text{-}\overline{\text{DD}}^n$ -property, we can establish the following general result implying Theorems 1, 2 and 3.

Theorem 4. *Let $p : K \rightarrow M$ be a perfect map defined on a paracompact submetrizable space K . If a subspace X of a completely-metrizable space Y possesses the $m\text{-}\overline{\text{DD}}^n$ -property for $m = \dim M$ and $n = \dim_{\Delta}(p)$, then*

$$\mathcal{E}(p, Y) = \{f \in C(K, Y) : p \triangle f : K \rightarrow M \times Y \text{ is an embedding}\}$$

is a G_{δ} -set in $C(K, Y)$ whose closure $\overline{\mathcal{E}(p, Y)}$ contains all simplicially factorizable maps from K to X . More precisely, for any continuous pseudometric ρ on Y , a continuous function $\varepsilon : K \rightarrow (0, +\infty)$ and a simplicially factorizable map $f : K \rightarrow X$ there is a map $g \in \mathcal{E}(p, Y)$ and an ε -homotopy $h : K \times [0, 1] \rightarrow Y$ connecting f and g so that $h(K \times [0, 1]) \subset X$.

A map $f : K \rightarrow X$ is called *simplicially factorizable* if there exist a simplicial complex L and maps $\alpha : K \rightarrow L$ and $\beta : L \rightarrow X$ such that $f = \beta \circ \alpha$. It turns out that in many important cases simplicially factorizable maps form a dense set in the function space $C(K, X)$. To describe such cases, we need the notion of a Lefschetz ANE[n]-space that is a parameterized version of a space satisfying the Lefschetz condition, see [Bors, V.8].

Let \mathcal{U} be a cover of a space X and K be a simplicial complex. By a *partial \mathcal{U} -realization* of K in X we understand any continuous map $f : L \rightarrow X$ defined on a geometric subcomplex $L \subset K$ containing all vertices of K and such that $\text{diam} f(\sigma \cap L) < \mathcal{U}$ for every simplex σ of K . If $L = K$, then the map f is called a *full \mathcal{U} -realization* of K in X .

A topological space X is defined to be a *Lefschetz ANE[n]* if for every open cover \mathcal{U} of X there is an open cover \mathcal{V} of X such that each partial \mathcal{V} -realization $f : L \rightarrow X$ of a simplicial complex K with $\dim K \leq n$ can be extended to a full \mathcal{U} -realization $\tilde{f} : K \rightarrow X$ of K .

Lefschetz ANE[n]-spaces are tightly connected with both ANR's and LC^n -spaces and have all basic properties of such spaces.

Proposition 3. *Let n be a non-negative integer or infinity.*

- (1) *A metrizable space X is a Lefschetz ANE[n] if and only if X is an ANE[n] for the class of metrizable spaces;*
- (2) *If n is finite, then a regular (paracompact) space X is a Lefschetz ANE[n] (if and) only if X is LC^{n-1} ;*
- (3) *Each convex subset X of a (locally convex) linear topological space L is a Lefschetz ANE[n] for any finite n (is a Lefschetz ANE[∞]);*
- (4) *There exists a metrizable σ -compact linear topological space that fails to be a Lefschetz ANE[∞];*
- (5) *A neighborhood retract of a Lefschetz ANE[n]-space is a Lefschetz ANE[n]-space;*
- (6) *A functionally open subspace of a Lefschetz ANE[n] is a Lefschetz ANE[n];*
- (7) *A topological space X is a Lefschetz ANE[n] if X has a uniform open cover by Lefschetz ANE[n]-spaces;*
- (8) *A metric space (X, ρ) is a Lefschetz ANE[n] if for every $\varepsilon > 0$ there is $\delta > 0$ such that each partial $\mathcal{B}_\rho(\delta)$ -realization $f : L \rightarrow X$ of a simplicial complex K with $\dim K \leq n$ extends to a full $\mathcal{B}_\rho(\varepsilon)$ -realization $\tilde{f} : K \rightarrow X$ of K in X ;*
- (9) *For each continuous pseudometric η on a paracompact Lefschetz ANE[n]-space X there is a continuous pseudometric $\rho \geq \eta$ such that for every $r \in (0, 1/2]$ each partial $\mathcal{D}_\rho(r/8)$ -realization $f : L \rightarrow X$ of a simplicial complex K with $\dim K \leq n$ extends to a full $\mathcal{D}_\rho(r)$ -realization $\tilde{f} : K \rightarrow X$ of K in X ;*
- (10) *Each map $f : X \rightarrow Y$ from a paracompact Lefschetz ANE[n]-space to a metrizable space Y factorizes through a metrizable Lefschetz ANE[n]-space Z in the sense that $f = g \circ h$ for some maps $h : X \rightarrow Z$ and $g : Z \rightarrow Y$.*

Here by $\mathcal{D}_\rho(\varepsilon)$ we denote the cover of a metric space (X, ρ) by all open sets of diameter $< \varepsilon$. With the notion of Lefschetz ANE[n]-space at hands, we can returns to simplicially factorizable maps.

Proposition 4. *Simplicially factorizable maps from a paracompact space K into a Tychonoff space X form a dense set in the function space $C(K, X)$ if one of the following conditions is satisfied:*

- (1) X is a Lefschetz ANE $[k]$ for $k = \dim K$;
- (2) K is a C -space and X is a locally contractible paracompact space.

Observe that Theorem 1, 2, and 3 follow immediately from Theorem 4 and Propositions 4, 3 and 1.

Combining Theorem 4 with Propositions 4(1), 3(2) and 1(4), we obtain another generalization of Theorem 1.

Theorem 5. *Let $p : K \rightarrow M$ be a perfect map between finite-dimensional paracompact spaces with K being submetrizable. If X is a completely metrizable LC^{k-1} -space possessing the $m\text{-}\overline{\text{DD}}^n$ -property, where $k = \dim K$, $m = \dim M$ and $n = \dim(p)$, then the function space $C(K, X)$ contains a dense G_δ -set of maps injective on each fiber of p .*

1.4. Approximating perfect maps by perfect PL-maps. The proof of Theorem 4 heavily exploits the technique of approximations by PL-maps. By a *PL-map* (resp., a *simplicial map*) we understand a map $f : K \rightarrow M$ between simplicial complexes which maps each simplex σ of K into (resp., onto) some simplex τ of M and f is linear on σ .

Theorem 6. *If $p : X \rightarrow Y$ is a perfect map between paracompact spaces, then for any open cover \mathcal{U} of X there exists an open cover \mathcal{V} of Y such that for any \mathcal{V} -map $\beta : Y \rightarrow M$ into a simplicial complex M there are an \mathcal{U} -map $\alpha : X \rightarrow K$ into a simplicial complex K and a perfect PL-map $f : K \rightarrow M$ with $f \circ \alpha = \beta \circ p$ and $\dim_\Delta(f) = \dim(f) \leq \dim_\Delta(p)$.*

Since for each open cover \mathcal{V} of a paracompact space Y there is a \mathcal{V} -map $\beta : Y \rightarrow M$ into a simplicial complex of dimension $\dim M \leq \dim Y$, Theorem 6 implies the following approximation result.

Corollary 1. *If $p : X \rightarrow Y$ is a perfect map between paracompact spaces, then for any open covers \mathcal{U} and \mathcal{V} of X and Y , respectively, there exist a \mathcal{U} -map $\alpha : X \rightarrow K$ into a simplicial complex K of dimension $\dim K \leq \dim Y + \dim_\Delta(p)$, a \mathcal{V} -map $\beta : Y \rightarrow M$ to a simplicial complex M of dimension $\dim M \leq \dim Y$, and a perfect PL-map $f : K \rightarrow M$ of dimension $\dim(f) \leq \dim_\Delta(p)$ making the following diagram commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & K \\ p \downarrow & & \downarrow f \\ Y & \xrightarrow{\beta} & M \end{array}$$

For light maps $p : X \rightarrow Y$ between metrizable compacta this corollary was proved by A.Dranishnikov and V.Uspenskij in [DU] and for arbitrary maps between metrizable compacta by Yu.Turygin [Tu].

1.5. $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties. Because of the presence of the $m\text{-}\overline{\text{DD}}^n$ -property in Theorems 4–5, it is important to have convenient methods for detecting that property. To establish such methods, we introduce the following three-parametric version of $m\text{-}\overline{\text{DD}}^n$.

Definition 2. A space X is defined to have the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if for any open cover \mathcal{U} of X and two maps $f : \mathbb{I}^m \times \mathbb{I}^n \rightarrow X$, $g : \mathbb{I}^m \times \mathbb{I}^k \rightarrow X$ there exist maps $f' : \mathbb{I}^m \times \mathbb{I}^n \rightarrow X$, $g' : \mathbb{I}^m \times \mathbb{I}^k \rightarrow X$ such that $f' \underset{\mathcal{U}}{\sim} f$, $g' \underset{\mathcal{U}}{\sim} g$, and $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in \mathbb{I}^m$.

By $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ we shall denote the class of all spaces with the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property. It is clear that the $m\text{-}\overline{\text{DD}}^n$ -property coincides with the $m\text{-}\overline{\text{DD}}^{\{n,n\}}$ -property. If some of the numbers m, n, k are infinite, the detection of the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property can be reduced to the detection of $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ with finite m, n, k .

Proposition 5. A Tychonoff space X has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if and only if it has the $a\text{-}\overline{\text{DD}}^{\{b,c\}}$ -property for all $a < m + 1$, $b < n + 1$, $c < k + 1$.

The proof of Theorem 4 is based on the following simplicial characterization of the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property.

Theorem 7. A submetrizable space X has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if and only if for any

- simplicial maps $p_N : N \rightarrow M$, $p_K : K \rightarrow M$ between finite simplicial complexes with $\dim M \leq m$, $\dim(p_N) \leq n$, $\dim(p_K) \leq k$,
- open cover \mathcal{U} of X , and
- maps $f : N \rightarrow X$, $g : K \rightarrow X$,

there exist maps $f' : N \rightarrow X$, $g' : K \rightarrow X$ such that $f' \underset{\mathcal{U}}{\sim} f$, $g' \underset{\mathcal{U}}{\sim} g$ and, for every $z \in M$ we have $f'(p_N^{-1}(z)) \cap g'(p_K^{-1}(z)) = \emptyset$.

Using the above simplicial characterization, we can establish the local nature of the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property.

Proposition 6. Let m, n, k be non-negative integers or ∞ .

- (1) If a space X has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property, then each open subspace of X also has that property.
- (2) A paracompact submetrizable space X has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if and only if it admits a cover by open subspaces with that property.

The $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property is also preserved by taking homotopically n -dense subspaces. We define a subset A of a topological space X to be *homotopically n -dense* in X if the following conditions are satisfied:

- for every map $f : \mathbb{I}^n \rightarrow X$ and an open cover \mathcal{U} of X there is a map $f' : \mathbb{I}^n \rightarrow A$ that is \mathcal{U} -homotopic to f ;
- for every open cover \mathcal{U} of X there is an open cover \mathcal{V} of X such that if two maps $f, g : \mathbb{I}^n \rightarrow A$ are \mathcal{V} -homotopic in X , then they are \mathcal{U} -homotopic in A .

By Theorem 2.8 of [To1], each dense relative LC^n -subset X of a metrizable space \tilde{X} is homotopically n -dense in \tilde{X} . The following useful proposition follows immediately from the definitions and the mentioned theorem of Toruńczyk.

Proposition 7. A homotopically $\max\{m+n, m+k\}$ -dense subspace X of a topological space \tilde{X} has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if and only if \tilde{X} has that property. Consequently, a dense relative $\text{LC}^{m+\max\{n,k\}}$ -set X in a space \tilde{X} has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if and only if \tilde{X} has that property.

This fact will be often applied in combination with Proposition 2.8 from [DM] asserting that each metrizable LC^n -space X embeds into a completely metrizable LC^n -space \tilde{X} as a dense relative LC^n -set. The last assertion enable us to apply Baire Category arguments for establishing the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties in arbitrary (not necessary complete) metric spaces.

Next, we elaborate tools for detecting the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties. Recall that a space X has no free arcs if X contains no open subset, homeomorphic to a non-empty connected subset of the real line. In particular, a space without free arcs has no isolated points.

- Proposition 8.** (1) A topological space X has the $0\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property if and only if each path-connected component of X is non-degenerate.
(2) An LC^0 -space X has the $0\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property if and only if X has no isolated point.
(3) A metrizable LC^1 -space X has the $0\text{-}\overline{\text{DD}}^{\{0,1\}}$ -property iff X has the $1\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property iff X has no free arc.
(4) Any metrizable LC^n -space X with the $0\text{-}\overline{\text{DD}}^{\{0,n\}}$ -property and $\dim X \leq n$ has the $0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property.
(5) A Polish $\text{ANE}[\max\{n,k\} + 1]$ -space X has the $0\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if and only if there are two disjoint dense σ -compact subsets A, B of X such that A is relative LC^{n-1} and B is relative LC^{k-1} in X .

Items 3 and 4 of Proposition 8 imply that each one dimensional LC^1 -space without free arcs has the $0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property. In particular, each dendrite with a dense set of end-points has that property.

The last item of Proposition 8 is a partial case of a more general characterization of the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property in terms of mapping absorption properties.

Let M, X be topological spaces. We shall say that a subset $A \subset M \times X$ has the *absorption property for n -dimensional maps in M* (briefly, $M\text{-MAP}^n$) if for any n -dimensional map $p : K \rightarrow M$ with K being a finite-dimensional compact space, a closed subset $C \subset K$, a map $f : K \rightarrow X$, and an open cover \mathcal{U} of X there is a map $f' : K \rightarrow X$ such that f' is \mathcal{U} -homotopic to f , $f'|_C = f|_C$ and $(p \triangle f')(K \setminus C) \subset A$. If $M = \mathbb{I}^m$, then we write $m\text{-MAP}^n$ instead of $\mathbb{I}^m\text{-MAP}^n$.

Theorem 8. Let m, n, k be non-negative integers or infinity and $d = 1 + m + \max\{n, k\}$. A (Polish $\text{ANE}[d]$ -)space X has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if (and only if) for any separable polyhedron M with $\dim M \leq m$ there are two disjoint (σ -compact) sets $E, F \subset M \times X$ such that E has $M\text{-MAP}^n$ and F has $M\text{-MAP}^k$.

Let us observe that the existence of such disjoint sets A, B is not obvious even for a dendrite with a dense set of end-points. Such a dendrite D has the $1\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property and thus the product $\mathbb{I} \times D$ contains two disjoint σ -compact subsets with 1-MAP^0 .

1.6. A Selection Theorem for Z_n -set-valued functions. Many results on $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties are based on a selection theorem for Z_n -valued functions, discussed in this subsection.

A subset A of a topological space X is called a (*homotopical*) Z_n -set in X if A is closed in X and for any an open cover \mathcal{U} of X and a map $f : \mathbb{I}^n \rightarrow X$ there is a map $g : \mathbb{I}^n \rightarrow X$ such that $g(\mathbb{I}^n) \cap A = \emptyset$ and g is \mathcal{U} -near (\mathcal{U} -homotopic) to f .

Each homotopical Z_n -set in a topological space X is a Z_n -set in X . The converse is true if X is an LC^n -space, see Theorem 15.

A set-valued function $\Phi : X \Rightarrow Y$ is defined to be *compactly semi-continuous* if for every compact subset $K \subset Y$ the preimage $\Phi^{-1}(K) = \{x \in X : \Phi(x) \cap K \neq \emptyset\}$ is closed in X .

Theorem 9. *Let $\Phi : X \Rightarrow Y$ be a compactly semi-continuous set-valued function from a paracompact C -space X into a topological space Y , assigning to each point $x \in X$ a homotopical Z_n -set $\Phi(x)$, where $n = \dim X \leq \infty$. If X is a retract of an open subset of a locally convex linear topological space, then for any map $f : X \rightarrow Y$ and any continuous pseudometric ρ on Y there is map $f' : X \rightarrow Y$ such that $f'(x) \notin \Phi(x)$ for all $x \in X$ and f' is 1-homotopic to f with respect to ρ .*

In particular, the theorem is true for stratifiable ANR's X (which are neighborhood retracts of stratifiable locally convex spaces, see [Si]).

1.7. Homotopical Z_n -sets and $m\text{-}\overline{DD}^{\{n,k\}}$ -properties. It turns out that homotopical Z_n -sets are tightly connected with the $m\text{-}\overline{DD}^{\{n,k\}}$ -properties. A point x of a topological space X is called a (*homotopical*) Z_n -point if the singleton $\{x\}$ is a (homotopical) Z_n -set in X . By $\mathcal{Z}_n(X)$ we shall denote the set of all homotopical Z_n -points of a space X .

Let

- \mathcal{Z}_n be the class of Tychonoff spaces X with $\mathcal{Z}_n(X) = X$;
- $\overline{\mathcal{Z}}_n$ be the class of Tychonoff spaces X with $\overline{\mathcal{Z}}_n(X) = X$;
- $\Delta\mathcal{Z}_n$ be the class of Tychonoff spaces X whose diagonal Δ_X is a homotopical Z_n -set in X^2 ;

For example, $\mathbb{R}^{n+1} \in \mathcal{Z}_n \cap \Delta\mathcal{Z}_n$.

Besides the classes of spaces related to Z -sets, we also need some other (more familiar) classes of topological spaces:

- Br , the class of metrizable separable Baire spaces,
- Π_2^0 , the class of Polish spaces, and
- LC^n , the class of all LC^n -spaces.

Theorem 10. *Let m, n, k be non-negative integers or infinity.*

- (1) *A space X has the $n\text{-}\overline{DD}^{\{0,0\}}$ -property if and only if the diagonal of X^2 is a homotopical Z_n -set in X^2 . This can be written as*

$$\Delta\mathcal{Z}_n = n\text{-}\overline{DD}^{\{0,0\}}$$

- (2) *An LC^0 -space X has the $0\text{-}\overline{DD}^{\{0,n\}}$ -property provided the set $\mathcal{Z}_n(X)$ is dense in X . This can be written as:*

$$LC^0 \cap \overline{\mathcal{Z}}_n \subset 0\text{-}\overline{DD}^{\{0,n\}}$$

- (3) *If a metrizable separable Baire (LC^n -)space X has the $0\text{-}\overline{DD}^{\{0,n\}}$ -property then the set of (homotopical) Z_n -points is a dense G_δ -set in X :*

$$\text{Br} \cap LC^n \cap 0\text{-}\overline{DD}^{\{0,n\}} \subset \overline{\mathcal{Z}}_n$$

- (4) *If each point of a space X is a homotopical Z_{m+k} -point, then X has the $m\text{-}\overline{DD}^{\{0,k\}}$ -property:*

$$\mathcal{Z}_{m+k} \subset m\text{-}\overline{\text{DD}}^{\{0,k\}}$$

- (5) If a topological space X has either the $n\text{-}\overline{\text{DD}}^{\{n,0\}}$ - or the $0\text{-}\overline{\text{DD}}^{\{n,n\}}$ -property, then each point of X is a homotopical Z_n -point:

$$0\text{-}\overline{\text{DD}}^{\{n,n\}} \cup n\text{-}\overline{\text{DD}}^{\{n,0\}} \subset \mathcal{Z}_n$$

- (6) If a Tychonoff space X has the $2\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property, then each point of X is a homotopical Z_1 -point:

$$2\text{-}\overline{\text{DD}}^{\{0,0\}} \subset \mathcal{Z}_1$$

1.8. Arithmetics of $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties. In this subsection we study the behavior of the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties under arithmetic operations. The combination of the results from this subsection and Propositions 6–8 provides convenient tools for detecting the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties of more complex spaces (like products or manifolds).

For a better visual presentation of our subsequent results, let us introduce the following operations on subclasses $\mathcal{A}, \mathcal{B} \subset \text{Top}$ of the class Top of topological spaces:

$$\mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\},$$

$$\frac{\mathcal{A}}{\mathcal{B}} = \{X \in \text{Top} : \exists B \in \mathcal{B} \text{ with } X \times B \in \mathcal{A}\},$$

$$\mathcal{A}^k = \{A^k : A \in \mathcal{A}\} \text{ and } \sqrt[k]{\mathcal{A}} = \{A \in \text{Top} : A^k \in \mathcal{A}\}.$$

A space X will be identified with the one-element class $\{X\}$. So $X \times \mathcal{A}$ and $\frac{\mathcal{A}}{X}$ means $\{X\} \times \mathcal{A}$ and $\frac{\mathcal{A}}{\{X\}}$.

We recall that $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ stands for the class of all spaces possessing the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property and LC^n is the class of LC^n -spaces.

Theorem 11 (Multiplication Formulas). *Let X, Y be metrizable spaces and $k_1, k_2, k, n_1, n_2, n, m_1, m_2, m$ be non-negative integers or infinity.*

- (1) **(First Multiplication Formula)**

If X has the $m\text{-}\overline{\text{DD}}^{\{n,k_1\}}$ -property and Y has the $m\text{-}\overline{\text{DD}}^{\{n,k_2\}}$ -property, then the product $X \times Y$ has the $m\text{-}\overline{\text{DD}}^{\{n,k_1+k_2+1\}}$ -property. This can be written as

$$m\text{-}\overline{\text{DD}}^{\{n,k_1\}} \times m\text{-}\overline{\text{DD}}^{\{n,k_2\}} \subset m\text{-}\overline{\text{DD}}^{\{n,k_1+k_2+1\}}$$

- (2) **(Second Multiplication Formula)**

If X has the $m\text{-}\overline{\text{DD}}^{\{n_1,k_1\}}$ -property and Y has both the $m\text{-}\overline{\text{DD}}^{\{n,k_2\}}$ - and $m\text{-}\overline{\text{DD}}^{\{n_2,k\}}$ -properties, where $n = n_1 + n_2 + 1$ and $k = k_1 + k_2 + 1$, then the product $X \times Y$ has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property. This can be written as

$$m\text{-}\overline{\text{DD}}^{\{n_1,k_1\}} \times (m\text{-}\overline{\text{DD}}^{\{n,k_2\}} \cap m\text{-}\overline{\text{DD}}^{\{n_2,k\}}) \subset m\text{-}\overline{\text{DD}}^{\{n,k\}}$$

- (3) **(Multiplication by a cell)**

If X has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property, then for any $d < m+1$ the product $\mathbb{I}^d \times X$ has the $(m-d)\text{-}\overline{\text{DD}}^{\{d+n,d+k\}}$ -property. This can be written as

$$\mathbb{I}^d \times m\text{-}\overline{\text{DD}}^{\{n,k\}} \subset (m-d)\text{-}\overline{\text{DD}}^{\{d+n,d+k\}}$$

Remark 1. Let us mention that, since $\mathbb{R} \in 0\text{-}\overline{\text{DD}}^{\{0,0\}}$, the second multiplication formula implies the following result of W. Mitchell [WM, Theorem 4.3(3)] (see also R. Daverman [Da1, Proposition 2.8]): If X is a compact metric ANR-space with $X \in 0\text{-}\overline{\text{DD}}^{\{p,p+1\}}$, then $X \times \mathbb{R} \in 0\text{-}\overline{\text{DD}}^{\{p+1,p+1\}}$. Moreover, Theorem 11(2) yields $X \times \mathbb{R}^{m+p+1} \in m\text{-}\overline{\text{DD}}^{\{n+p+1,k\}}$ for any metrizable space $X \in m\text{-}\overline{\text{DD}}^{\{n,k\}} \cap m\text{-}\overline{\text{DD}}^{\{n+p+1,k-1\}}$. The partial case of this result when $m = 0$ and $p = 1$ was provided by W. Mitchell in [WM, Theorem 4.3(2)]. Similarly, we can see that Theorem 11(3) generalizes the following result of D. Halverson [Hal]: If X is a locally compact ANR with $X \in 1\text{-}\overline{\text{DD}}^{\{1,1\}}$, then $X \times \mathbb{R} \in 0\text{-}\overline{\text{DD}}^{\{2,2\}}$.

Next, we consider the so-called base enlargement formulas expressing the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property via $0\text{-}\overline{\text{DD}}^{\{n',k'\}}$ -properties for sufficiently large n', k' .

Theorem 12 (Base Enlargement Formulas). *Let X be a metrizable space and n, k, m, m_1, m_2 be non-negative integers or infinity.*

- (1) *If X possesses the $0\text{-}\overline{\text{DD}}^{\{n+m_1,k+m_2\}}$ -, $m_1\text{-}\overline{\text{DD}}^{\{n,k+m-m_1\}}$ -, and $m_2\text{-}\overline{\text{DD}}^{\{n+m-m_2,k\}}$ -properties simultaneously with $m = m_1 + m_2 + 1$, then X has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property. This can be written as*

$$0\text{-}\overline{\text{DD}}^{\{n+m_1,k+m_2\}} \cap m_1\text{-}\overline{\text{DD}}^{\{n,k+m-m_1\}} \cap m_2\text{-}\overline{\text{DD}}^{\{n+m-m_2,k\}} \subset m\text{-}\overline{\text{DD}}^{\{n,k\}}$$

- (2) *If $X \in 0\text{-}\overline{\text{DD}}^{\{n,k+m+1\}} \cap m\text{-}\overline{\text{DD}}^{\{n+1,k\}}$, then X has the $(m+1)\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property. This can be written as*

$$0\text{-}\overline{\text{DD}}^{\{n,k+m+1\}} \cap m\text{-}\overline{\text{DD}}^{\{n+1,k\}} \subset (m+1)\text{-}\overline{\text{DD}}^{\{n,k\}}$$

- (3) *X has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property if X has the $0\text{-}\overline{\text{DD}}^{\{n+i,k+j\}}$ -property for all $i, j \in \omega$ with $i + j < m + 1$. This can be written as*

$$\bigcap_{i+j < m+1} 0\text{-}\overline{\text{DD}}^{\{n+i,k+j\}} \subset m\text{-}\overline{\text{DD}}^{\{n,k\}}$$

The second base enlargement formula implies that if X is a metrizable space with $X \in 0\text{-}\overline{\text{DD}}^{\{1,2\}}$, then $X \in 1\text{-}\overline{\text{DD}}^{\{1,1\}}$. This result was established by D. Halverson [Hal] in the particular case when X is a separable locally compact ANR.

1.9. $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties of products. In this subsection we apply the arithmetic formulas from previous subsection to establish the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties of products.

Theorem 13. *Let m, n, k, d, l be non-negative integers and L, D be metrizable spaces such that L has the $0\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property and D has the $0\text{-}\overline{\text{DD}}^{\{0,d+l\}}$ -property. If $m + n + k < 2d + l$, then the product $D^d \times L^l$ has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property. This can be written as*

$$(0\text{-}\overline{\text{DD}}^{\{0,0\}})^l \times (0\text{-}\overline{\text{DD}}^{\{0,d+l\}})^d \subset \bigcap_{m+n+k < 2d+l} m\text{-}\overline{\text{DD}}^{\{n,k\}}$$

Combining Theorem 13 with Theorem 4, Proposition 2 and Proposition 8 we obtain

Theorem 14. *Let l, d be non-negative integers or infinity and L, D be completely metrizable locally path-connected spaces such that L has no isolated points and D is*

1-dimensional without free arcs. Then the product $D^d \times L^l$ has the $m\text{-}\overline{\text{DD}}^n$ -property for all $m, n \in \omega$ with $m + 2n < l + 2d$. Consequently, if $p : K \rightarrow M$ is a perfect map between paracompact submetrizable spaces with $\dim M + 2\dim_{\Delta}(p) < l + 2d$, then any simplicially factorizable map $f : K \rightarrow D^d \times L^l$ can be approximated by maps injective on each fiber of p .

The case $m = 0$ from Theorem 14 yields

Corollary 2. *Let L, D be completely metrizable ANR's such that L has no isolated points and D is 1-dimensional without free arcs. Then the product $D^d \times L^l$ has the DD^nP for all $n < d + \frac{l}{2}$. Consequently, for any compact space X of dimension $\dim X < d + \frac{l}{2}$ the set of all embeddings is dense G_{δ} in the function space $C(X, D^d \times L^l)$.*

Remark 2. Corollary 2 generalizes many (if not all) results on embeddings into products. Indeed, letting $L = \mathbb{R}$ to be the real line and D to be a dendrite with a dense set of end-points we obtain the following well known results:

- the case $d = 0$ and $l = 2n + 1$ is the Lefschetz-Menger-Nöbeling-Pontrjagin embedding theorem that \mathbb{R}^{2n+1} has DD^nP ;
- the case $d = n + 1$ and $l = 0$ is the embedding theorem of P. Bowers [Bow] that D^{n+1} has DD^nP ;
- the case $d = n$ and $l = 1$ is the embedding theorem of Y. Sternfeld [Ste] that $D^n \times \mathbb{I}$ has DD^nP ;

Also, for $d = 0$ and $m = 0$ Theorem 14 is close to the embedding theorem from T. Banach, Kh. Trushchak [BTr] while for $l = 0$ and $m = 0$ it is close to that one of T. Banach, R. Cauty, Kh. Trushchak, L. Zdomskyy [BCTZ].

Remark 3. Letting $d = 0$ and $L = \mathbb{R}$ in Theorem 14, we obtain the Pasynkov's result [Pa1] asserting that for a map $p : X \rightarrow Y$ between finite-dimensional metrizable compacta the function space $C(X, \mathbb{R}^{\dim Y + 2\dim(p)+1})$ contains a dense G_{δ} -set of maps that are injective on each fiber of the map p .

Therefore, Theorem 14 can be considered as a generalization of [Pa1]. However, Theorem 14 does not cover another generalization of the Pasynkov's result due to H. Toruńczyk [To4]: *If $p : X \rightarrow Y$ is a map between compacta, then the space $C(X, \mathbb{R}^{\dim X + \dim(p)+1})$ contains a dense G_{δ} -set of maps that are injective on each fiber of the map p .*

Taking into account that the Euclidean space \mathbb{R}^d has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties for all m, n, k with $m + n + k < d$, we may ask whether the this theorem of H. Toruńczyk is true in the following more general form.

Problem 1. *Let $p : K \rightarrow M$ be a map between finite-dimensional compact metric spaces and X be a Polish AR-space possessing the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property for all m, n, k with $m + n + k \leq \dim K + \dim(p)$. Does p embed into the projection $\text{pr} : M \times X \rightarrow M$?*

Let us also note that the mentioned above result of H. Toruńczyk would follow from our Theorem 1 if the following problem had an affirmative answer.

Problem 2. *Let $f : X \rightarrow Y$ be a k -dimensional map between finite-dimensional metrizable compacta. Is it true that there is a map $g : Y \rightarrow Z$ to a compact space Z with $\dim Z \leq \dim X - k$ such that the map $g \circ f$ is still k -dimensional?*

1.10. A short survey on homological Z_n -sets. The most exciting results on $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties (like multiplication and k -root formulas) are obtained by using homological Z_n -sets. In this subsection we survey some basic facts about such sets, and refer the interested reader to [BCK] where all these results are established. We use the singular homology with coefficients in an Abelian group G . If $G = \mathbb{Z}$, we write $H_k(X)$ instead of $H_k(X; \mathbb{Z})$. By $\tilde{H}_*(X; G)$ we denote the singular homology of X reduced in dimension zero.

It can be shown that a closed subset A of a topological space X is a homotopical Z_n -set in X if and only if for every open set $U \subset X$ the inclusion $U \setminus A \rightarrow U$ is weak homotopy equivalence, which means that the relative homotopy groups $\pi_k(U, U \setminus A)$ vanish for all $k < n + 1$. Replacing the relative homotopy groups by relative homology groups, we obtain the notion of a homological Z_n -set.

A closed subset A of a space X is defined to be

- a G -homological Z_n -set in X for a coefficient group G if $H_k(U, U \setminus A; G) = 0$ for all open sets $U \subset X$ and all $k < n + 1$;
- an $\exists G$ -homological Z_n -set if A is a G -homological Z_n -set in X for some coefficient group G ;
- a homological Z_n -set if A is a \mathbb{Z} -homological Z_n -set in X (equivalently, if A is a G -homological Z_n -set for every coefficient group G).

In [DW] homological Z_∞ -sets are referred to as closed sets of infinite codimension. On the other hand, the term “homological Z_n -set” has been used in [JS], [BCK], [BC], and [BR].

The following theorem whose proof can be found in [BCK, Theorems 3.2-3.3] describes the interplay between various sorts of Z_n -sets.

Theorem 15. *Let X be a topological space.*

- (1) *Each homotopical Z_n -set in X is both a Z_n -set and a homological Z_n -set;*
- (2) *Each Z_n -set in an LC^n -space is a homotopical Z_n -set;*
- (3) *A set is a homotopical Z_0 -set in X iff it is a $\exists G$ -homological Z_0 -set;*
- (4) *Each $\exists G$ -homological Z_1 -set in X is a Z_1 -set;*
- (5) *If X is an LC^1 -space, then a homotopical Z_2 -set in X is a homotopical Z_n -set if and only if it is a homological Z_n -set.*

The last item of this theorem has fundamental importance since it allows application of powerful tools of Algebraic Topology for studying homotopical Z_n -sets and related $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties.

The study of G -homological Z_n -sets for an arbitrary group G can be reduced to considering Bockstein groups. Under the notations below,

- \mathbb{Q} - the group of rational numbers;
- $\mathbb{Z}_p = \mathbb{Z}/\mathbb{Z}_p$ - the cyclic group of a prime order p ;
- $\mathbb{Q}_p = \{z \in \mathbb{C} : \exists k \in \mathbb{N} \ z^{p^k} = 1\}$ - the quasicyclic p -group;
- $R_p = \{\frac{m}{n} : m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \text{ is not divisible by } p\}$,

the Bockstein family $\sigma(G)$ of a group G is a subfamily of $\{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, R_p : p \in \Pi\}$, where Π is the set of prime numbers, such that

- $\mathbb{Q} \in \sigma(G)$ iff $G/\text{Tor}(G)$ is divisible;
- $\mathbb{Z}_p \in \sigma(G)$ iff the p -torsion part $p\text{-Tor}(G)$ is not divisible by p ;
- $\mathbb{Q}_p \in \sigma(G)$ iff $p\text{-Tor}(G) \neq 0$ is divisible by p ;
- $R_p \in \sigma(G)$ iff the group $G/p\text{-Tor}(G)$ is not divisible by p .

Here

$\text{Tor}(G) = \{x \in G : \exists n \in \mathbb{N} \, n \cdot x = 0\}$ and $p\text{-Tor}(G) = \{x \in G : \exists k \in \mathbb{Z} \, p^k \cdot x = 0\}$ is the torsion and p -torsion parts of G . In particular, $\sigma(\mathbb{Z}) = \{R_p : p \in \Pi\}$.

Theorem 16. *Let A be a closed subset of a space X , G be a coefficient group, and p be a prime number.*

- (1) *A is a G -homological Z_n -set in X if and only if A is an H -homological Z_n -set in X for all groups $H \in \sigma(G)$.*
- (2) *If A is a R_p -homological Z_n -set in X , then A is a \mathbb{Q} -homological and \mathbb{Z}_p -homological Z_n -set in X .*
- (3) *If A is a \mathbb{Z}_p -homological Z_n -set in X , then A is a \mathbb{Q}_p -homological Z_n -set in X .*
- (4) *If A is a \mathbb{Q}_p -homological Z_{n+1} -set in X , then A is a \mathbb{Z}_p -homological Z_n -set in X .*
- (5) *A is a R_p -homological Z_n -set in X provided A is a \mathbb{Q} -homological Z_n -set in X and a \mathbb{Q}_p -homological Z_{n+1} -set in X .*

By analogy with multiplication formulas for the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties, there are multiplication formulas for homotopical and homological Z_n -sets, see [BCK, Theorem 6.1]).

Theorem 17. *Let $A \subset X$, $B \subset Y$ be closed subsets in Tychonoff spaces X, Y .*

- (1) *If A is a homotopical Z_n -set in X and B is a homotopical Z_m -set in X then $A \times B$ is a homotopical Z_{n+m+1} -set in $X \times Y$;*
- (2) *If A is a homological Z_n -set in X and B is a homological Z_m -set in X then $A \times B$ is a homological Z_{n+m+1} -set in $X \times Y$.*

Surprisingly, the multiplication formulas for homological Z_n -sets can be reversed:

Theorem 18. *Let $n, m \in \omega \cup \{\infty\}$, $k \in \omega$, and $A \subset X$, $B \subset Y$ be closed subsets of the spaces X and Y . Let $\mathfrak{D} = \{\mathbb{Q}, \mathbb{Q}_p : p \in \Pi\}$ and for every group $G \in \mathfrak{D}$ let $B_G \subset Y$ be a closed subset which fails to be a G -homological Z_m -set in Y . Then we have:*

- (1) *A is a homological Z_n -set in X if and only if A^k is a homological Z_{kn+k-1} -set in X^k ;*
- (2) *If $A \times B$ is an \mathbb{F} -homological Z_{n+m} -set in $X \times Y$ for some field \mathbb{F} , then either A is an \mathbb{F} -homological Z_n -set in X or B is an \mathbb{F} -homological Z_m -set in Y ;*
- (3) *If $A \times B$ is a homological Z_{n+m} -set in $X \times Y$, then either A is a homological Z_n -set in X or B is an $\exists G$ -homological Z_m -set in Y ;*
- (4) *A is a homological Z_n -set in X provided $A \times B_G$ is a homological Z_{n+m} -set in $X \times Y$ for every group $G \in \mathfrak{D}$.*

Theorem 18 is the principal tool for the proof of k -root and multiplication formulas about the classes $m\text{-}\overline{\text{DD}}^{\{n,k\}}$. We first discuss k -root and division formulas for the classes Z_n , \overline{Z}_n , and Z_n^Z because they are tightly connected with the classes $m\text{-}\overline{\text{DD}}^{\{n,k\}}$.

Let us start with some definitions. A point x of a space X is defined to be a *homological Z_n -point* if its singleton $\{x\}$ is a homological Z_n -set in X . By analogy, we define G -homological and $\exists G$ -homological Z_n -points.

Let $\mathcal{Z}_n^G(X)$ denote the set of all G -homological Z_n -points in a space X and $\overline{\mathcal{Z}}_n^G$ (resp., \mathcal{Z}_n^G) be the class of Tychonoff spaces X such that the set $\mathcal{Z}_n^G(X)$ is dense in (resp., coincides with) X . We also recall that $\overline{\mathcal{Z}}_n$ (resp., \mathcal{Z}_n) stands for the class of Tychonoff spaces X such that the set $\mathcal{Z}_n(X)$ of homotopical Z_n -points of X is dense in (resp., coincides with) X . Using these notations, Theorem 15 can be written in the following form.

Theorem 19. *Let $n \in \omega \cup \{\infty\}$ and G be a non-trivial Abelian group.*

- (1) $\mathcal{Z}_n \subset \mathcal{Z}_n^{\mathbb{Z}} \subset \mathcal{Z}_n^G$;
- (2) $\mathcal{Z}_0 = \mathcal{Z}_0^{\mathbb{Z}} = \mathcal{Z}_0^G$;
- (3) $\text{LC}^1 \cap \mathcal{Z}_1^G \subset \mathcal{Z}_1$;
- (4) $\text{LC}^1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_n^{\mathbb{Z}} \subset \mathcal{Z}_n$;
- (5) $\text{LC}^1 \cap \text{Br} \cap \overline{\mathcal{Z}}_2 \cap \overline{\mathcal{Z}}_n^{\mathbb{Z}} \subset \overline{\mathcal{Z}}_n$.

The last item of Theorem 19 follows from the fact that each of the sets $\mathcal{Z}_n(X)$ and $\mathcal{Z}_n^{\mathbb{Z}}(X)$ is G_δ in X provided X is a separable metrizable LC^n -space [BCK, Theorem 9.2].

In its turn, Theorem 17 implies multiplication formulas for the classes \mathcal{Z}_n , $\overline{\mathcal{Z}}_n$, and $\mathcal{Z}_n^{\mathbb{Z}}$:

Theorem 20 (Multiplication Formulas). *Let $n, m \in \omega \cup \{\infty\}$. Then*

- (1) $\mathcal{Z}_n \times \mathcal{Z}_m \subset \mathcal{Z}_{m+n+1}$;
- (2) $\mathcal{Z}_n^{\mathbb{Z}} \times \mathcal{Z}_m^{\mathbb{Z}} \subset \mathcal{Z}_{n+m+1}^{\mathbb{Z}}$;
- (3) $\overline{\mathcal{Z}}_n \times \overline{\mathcal{Z}}_m \subset \overline{\mathcal{Z}}_{m+n+1}$;
- (4) $\overline{\mathcal{Z}}_n^{\mathbb{Z}} \times \overline{\mathcal{Z}}_m^{\mathbb{Z}} \subset \overline{\mathcal{Z}}_{n+m+1}^{\mathbb{Z}}$.

The multiplication formulas can be reversed, which yields division and k -root formulas for the classes $\mathcal{Z}_n^{\mathbb{Z}}$ (we recall that, for a class \mathcal{A} , we put $\sqrt[k]{\mathcal{A}} = \{X : X^k \in \mathcal{A}\}$).

Theorem 21 (k -Root Formulas). *Let $n \in \omega \cup \{\infty\}$ and $k \in \mathbb{N}$.*

- (1) *A space X belongs to the class $\mathcal{Z}_n^{\mathbb{Z}}$ if and only if X^k belongs to $\mathcal{Z}_{kn+k-1}^{\mathbb{Z}}$:*

$$\mathcal{Z}_n^{\mathbb{Z}} = \sqrt[k]{\mathcal{Z}_{kn+k-1}^{\mathbb{Z}}}$$

- (2) *A metrizable separable Baire LC^{kn+k-1} -space X belongs to the class $\overline{\mathcal{Z}}_n^{\mathbb{Z}}$ if and only if X^k belongs to $\overline{\mathcal{Z}}_{kn+k-1}^{\mathbb{Z}}$:*

$$\overline{\mathcal{Z}}_n^{\mathbb{Z}} \supset \sqrt[k]{\overline{\mathcal{Z}}_{kn+k-1}^{\mathbb{Z}}} \cap \text{LC}^{kn+k-1} \cap \text{Br}$$

To state the division formula for the classes $\mathcal{Z}_n^{\mathbb{Z}}$ and $\overline{\mathcal{Z}}_n^{\mathbb{Z}}$ we need some more notations (which will be used for the classes $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ as well). Consider the following classes of topological spaces:

- $\cup_G \mathcal{Z}_n^G = \cup \{\mathcal{Z}_n^G : G \text{ is a non-trivial Abelian group}\}$;
- $\cup_G \overline{\mathcal{Z}}_n^G = \cup \{\overline{\mathcal{Z}}_n^G : G \text{ is a non-trivial Abelian group}\}$;
- $\exists_\circ \cup_G \overline{\mathcal{Z}}_n^G$, the class of spaces containing a non-empty open subspace $U \in \cup_G \overline{\mathcal{Z}}_n^G$.

For example, the space \mathbb{R}^n belongs to none of these classes.

Now, we can state the division formulas for the classes $\mathcal{Z}_n^{\mathbb{Z}}$ and $\overline{\mathcal{Z}}_n^{\mathbb{Z}}$ (recall that if \mathcal{A} and \mathcal{B} are two classes, then $\frac{\mathcal{A}}{\mathcal{B}}$ stands for the class $\{X \in \text{Top} : \exists B \in \mathcal{B} \text{ with } X \times B \in \mathcal{A}\}$).

Theorem 22 (Division Formulas). *Let $n \in \omega \cup \{\infty\}$ and $k \in \omega$.*

- (1) *A space X belongs to the class $\mathcal{Z}_n^{\mathbb{Z}}$ if and only if $X \times Y \in \mathcal{Z}_{n+k}^{\mathbb{Z}}$ for some space $Y \notin \cup_G \mathcal{Z}_k^G$. This can be written as*

$$\frac{\mathcal{Z}_{n+k}^{\mathbb{Z}}}{\text{Top} \setminus \cup_G \mathcal{Z}_k^G} = \mathcal{Z}_n^{\mathbb{Z}}$$

- (2) *A metrizable separable Baire LC^n -space X belongs to the class $\overline{\mathcal{Z}}_n^{\mathbb{Z}}$ if and only if $X \times Y \in \overline{\mathcal{Z}}_{n+k}^{\mathbb{Z}}$ for some space $Y \notin \exists_o \cup_G \overline{\mathcal{Z}}_k^G$. This can be written as*

$$\text{Br} \cap \text{LC}^n \cap \frac{\overline{\mathcal{Z}}_{n+k}^{\mathbb{Z}}}{\text{Top} \setminus \exists_o \cup_G \overline{\mathcal{Z}}_k^G} \subset \overline{\mathcal{Z}}_n^{\mathbb{Z}}$$

Because of the division formulas, it is important to detect the spaces $X \notin \cup_G \mathcal{Z}_n^G$. It turns out that this happens for every metrizable space X of dimension $\dim X \leq n$, or more generally of transfinite separation dimension $\text{trt}(X) < n + 1$. The latter dimension can be introduced inductively (see [ACP]):

- $\text{trt}(X) = -1$ iff $X = \emptyset$;
- $\text{trt}(X) \leq \alpha$ for an ordinal α if each closed subset $A \subset X$ with $|A| > 1$ contains a closed subset $B \subset A$ such that $\text{trt}(B) < \alpha$ and $A \setminus B$ is disconnected.

A space X is called *trt-dimensional* if $\text{trt}(X) \leq \alpha$ for some ordinal α . For a trt-dimensional space X we put $\text{trt}(X)$ be the smallest ordinal α with $\text{trt}(X) \leq \alpha$.

By [ACP], each compact metrizable trt-dimensional space is a C -space. On the other hand, a Čech-complete space is trt-dimensional if it can be written as the countable union of hereditarily disconnected subspaces, see [Ra]. It is easy to see that for a finite-dimensional metrizable separable space X we get $\text{trt}(X) \leq \dim(X)$. Moreover, if X is finite-dimensional and compact, then $\text{trt}(X) = \dim(X)$, see [St].

The following theorem was proved in [BCK] and [BC].

Theorem 23. *Let $X \in \cup_G \mathcal{Z}_n^G$ for some $n \in \omega \cup \{\infty\}$.*

- (1) *If $n < \infty$, then $\text{trt}(X) > n$;*
- (2) *If $n = \infty$, then X is not trt-dimensional;*
- (3) *If $n = \infty$ and X is locally compact and locally contractible, then X is not a C -space.*

Consequently, for any metrizable separable space $X \in \cup_G \mathcal{Z}_n^G$ we have $\dim X \geq \text{trt}(X) > n$. A similar inequality holds for cohomological and extension dimensions of X . We recall their definitions.

For a space X and a CW -complex L we write $\text{e-dim} X \leq L$ if each map $f : A \rightarrow L$ defined on a closed subset $A \subset X$ admits a continuous extension $\tilde{f} : X \rightarrow L$, see [DD] for more information on Extension Dimension Theory. It follows from the classical Hurewicz-Wallman Theorem [En1, 1.9.3] that $\text{e-dim} X \leq S^n$ iff $\dim X \leq n$.

The cohomological dimension with respect to a given Abelian group G can be expressed via extension dimension as follows: define $\dim_G X \leq n$ if $\text{e-dim} X \leq K(G, n)$, where $K(G, n)$ is the Eilenberg-MacLane complex of type (G, n) , and let $\dim_G X$ be the smallest non-negative integer with $\dim_G X \leq n$. If there is no such an integer n , we put $\dim_G X = \infty$.

Theorem 24. *Let $n \in \omega$ and $X \in \mathcal{Z}_n^{\mathbb{Z}}$ be a locally compact LC^n -space. Then*

- (1) $\dim_G X > n$ for any Abelian group G ;
- (2) $\text{e-dim} X \not\leq L$ for any CW-complex L with a non-trivial homotopy group $\pi_k(L)$ for some $k \leq n$.

1.11. Homological Z_n -sets and $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties. In this subsection we discuss the interplay between the classes \mathcal{Z}_n^G and $m\text{-}\overline{\text{DD}}^{\{n,k\}}$. The following two theorems present homological counterparts of the formulas

$$n\text{-}\overline{\text{DD}}^{\{n,0\}} \subset \mathcal{Z}_n \subset \bigcap_{m+k \leq n} m\text{-}\overline{\text{DD}}^{\{0,k\}} \text{ and}$$

$$\text{Br} \cap \text{LC}^n \cap 0\text{-}\overline{\text{DD}}^{\{0,n\}} \subset \text{LC}^0 \cap \overline{\mathcal{Z}}_n \subset 0\text{-}\overline{\text{DD}}^{\{0,n\}}$$

from Theorem 10.

Theorem 25. *Let X be a Tychonoff space and $n \in \omega \cup \{\infty\}$.*

- (1) *If an LC^1 -space X has the $2\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property, then each homological Z_n -point in X is a homotopical Z_n -point:*

$$\text{LC}^1 \cap 2\text{-}\overline{\text{DD}}^{\{0,2\}} \cap \mathcal{Z}_n^{\mathbb{Z}} \subset \mathcal{Z}_n$$

- (2) *If a metrizable separable Baire LC^n -space X has the $0\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property and contains a dense set of homological Z_n -points, then X contains a dense set of homotopical Z_n -points and $X \in 0\text{-}\overline{\text{DD}}^{\{0,n\}}$:*

$$\text{LC}^n \cap \text{Br} \cap 0\text{-}\overline{\text{DD}}^{\{0,2\}} \cap \overline{\mathcal{Z}}_n^{\mathbb{Z}} \subset \overline{\mathcal{Z}}_n$$

- (3) *If X has the $(2n+1)\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property, then each point of X is a homological Z_n -point:*

$$(2n+1)\text{-}\overline{\text{DD}}^{\{0,0\}} \subset \mathcal{Z}_n^{\mathbb{Z}}$$

- (4) *If X has the $2n\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property, then each point of X is a G -homological Z_n -point for any group G with divisible quotient $G/\text{Tor}(G)$. Consequently,*

$$2n\text{-}\overline{\text{DD}}^{\{0,0\}} \subset \mathcal{Z}_n^{\mathbb{Q}} \cap \bigcap_p (\mathcal{Z}_n^{\mathbb{Z}_p} \cap \mathcal{Z}_n^{\mathbb{Q}_p})$$

Theorem 26. *Let m, m, k be non-negative integers or infinity.*

- (1) *If each point of an LC^1 -space X with the $2\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property is a homological Z_{m+k} -point, then X has the $m\text{-}\overline{\text{DD}}^{\{0,k\}}$ -property. This can be written as*

$$\text{LC}^1 \cap \mathcal{Z}_{m+k}^{\mathbb{Z}} \cap 2\text{-}\overline{\text{DD}}^{\{0,2\}} \subset m\text{-}\overline{\text{DD}}^{\{0,k\}}$$

- (2) *If a metrizable separable LC^k -space X has the $0\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property and contains a dense set of homological Z_k -points, then X has the $0\text{-}\overline{\text{DD}}^{\{0,k\}}$ -property. This can be written as*

$$\boxed{\mathrm{LC}^k \cap \overline{\mathcal{Z}}_k^{\mathbb{Z}} \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \subset 0\text{-}\overline{\mathrm{DD}}^{\{0,k\}}}$$

- (3) If each point of a metrizable separable LC^2 -space X is a homological Z_{m+n+k} -point and X has the $m\text{-}\overline{\mathrm{DD}}^{\{n, \max\{n,2\}\}}$ -property, then X has the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$ -property. This can be written as

$$\boxed{\mathrm{LC}^2 \cap \mathcal{Z}_{m+n+k}^{\mathbb{Z}} \cap m\text{-}\overline{\mathrm{DD}}^{\{n, \max\{2,n\}\}} \subset m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

- (4) If each point of an LC^1 -space X is a homological Z_m -point and $X \in 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$, then X has the $m\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$ -property. This can be written as

$$\boxed{\mathrm{LC}^1 \cap \mathcal{Z}_m^{\mathbb{Z}} \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \subset m\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}$$

- (5) If each point of an LC^0 -space X is a G -homological Z_2 -point for some group G , then X has the $\mathrm{DD}^1\mathrm{P}$. If in addition X is a metrizable LC^1 -space containing a dense set of homotopical Z_2 -points, then $X \in 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$.

$$\boxed{\mathrm{LC}^0 \cap (\cup_G \mathcal{Z}_2^G) \subset 0\text{-}\overline{\mathrm{DD}}^{\{1,1\}} \quad \text{and} \quad \mathrm{LC}^1 \cap \overline{\mathcal{Z}}_2 \cap (\cup_G \mathcal{Z}_2^G) \subset 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}$$

1.12. Homological characterization of the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$ -property. In this subsection we prove a quantified version of the homological characterization of the $0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$ -property due to Daverman and Walsh [DW].

First, we provide a homotopical version of the Daverman-Walsh result.

Theorem 27. Let n, k be finite or infinite integers (with $n \leq k$). A Polish (LC^k) -space X has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$ -property if (and only if) there is a countable family \mathcal{F} of (n -dimensional compact) homotopical Z_k -sets in X such that each compact subset $K \subset X \setminus \cup \mathcal{F}$ is a homotopical Z_n -set in X .

Under some mild assumptions on X it is possible to replace the homotopical conditions in Theorem 27 by homological ones.

Theorem 28. A Polish $\mathrm{LC}^{\max\{n,k\}}$ -space $X \in 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$ has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$ -property provided each point of X is a homological $Z_{2+\max\{n,k\}}$ -point and there is a countable family \mathcal{F} of homological Z_k -sets in X such that each compact subset $K \subset X \setminus \cup \mathcal{F}$ is a homological Z_n -set in X .

This theorem implies another characterization of $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$ in terms of approximation properties defined as follows. We shall say that a topological space has the n -dimensional approximation property (briefly, $\mathrm{AP}[n]$) if for any open cover \mathcal{U} of X and a map $f : \mathbb{I}^n \rightarrow X$ there is a map $g : \mathbb{I}^n \rightarrow X$ such that g is \mathcal{U} -homotopic to f and $\mathrm{trt}(g(\mathbb{I}^n)) < n + 1$. Here we assume that $\alpha < \infty + 1$ for each ordinal α (which is essential if $n = \infty$).

Observe that each LC^0 -space has $\mathrm{AP}[0]$ and each LC^1 -space has $\mathrm{AP}[1]$.

Theorem 29. If each point of a Polish $\mathrm{LC}^{\max\{n,k\}}$ -space X is a homological Z_{n+k} -point and X has the properties $\mathrm{AP}[n]$ and $0\text{-}\overline{\mathrm{DD}}^{\{2, \min\{2,n\}\}}$, then X has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$ -property. This can be written as

$$\boxed{\Pi_2^0 \cap \mathrm{LC}^{\max\{n,k\}} \cap \mathcal{Z}_{n+k}^{\mathbb{Z}} \cap \mathrm{AP}[n] \cap 0\text{-}\overline{\mathrm{DD}}^{\{2, \min\{2,n\}\}} \subset 0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

1.13. $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties of locally rectifiable spaces. There is a non-trivial interplay between $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties for spaces having a kind of a homogeneity property. We recall that a space X is *topologically homogeneous* if for any two points $x_0, x \in X$ there is a homeomorphism $h_x : X \rightarrow X$ such that $h_x(x_0) = x$. If the homeomorphism h_x can be chosen to depend continuously on x then X is called *rectifiable* at x_0 .

More precisely, we define a topological space X to be *locally rectifiable at a point* $x_0 \in X$ if there exists a neighborhood U of x_0 such that for every $x \in U$ there is a homeomorphism $h_x : X \rightarrow X$ such that $h_x(x_0) = x$ and h_x continuously depends on x in the sense that the map $H : U \times X \rightarrow U \times X$, $H : (x, z) \mapsto (x, h_x(z))$ is a homeomorphism. If $U = X$, then the space X is called *rectifiable* at x_0 .

A space X is called *(locally) rectifiable* if it is (locally) rectifiable at each point $x \in X$. Rectifiable spaces were studied in details by A.S. Gulko [Gul]. The class of rectifiable spaces contains the underlying spaces of topological groups but also contains spaces not homeomorphic to topological groups. A simplest such an example is the 7-dimensional sphere S^7 , see [Us89]. It should be mentioned that all finite-dimensional spheres S^n are locally rectifiable but only S^1, S^3 and S^7 are rectifiable (this follows from the famous Adams' result [Ad] detecting H -spaces among the spheres). It can be shown that each connected locally rectifiable space is topologically homogeneous. On the other hand, the Hilbert cube is topologically homogeneous but fails to be (locally) rectifiable, see [Gul].

By \mathcal{LR} we denote the class of Tychonoff locally rectifiable spaces.

Theorem 30. *Let X be a locally rectifiable Tychonoff space.*

- (1) *If X has the $m\text{-}\overline{\text{DD}}^{\{0,k\}}$ -property, then each point of X is a homotopical Z_{m+k} -point:*

$$\mathcal{LR} \cap m\text{-}\overline{\text{DD}}^{\{0,k\}} \subset \mathcal{Z}_{m+k}$$

- (2) *If X has the $m\text{-}\overline{\text{DD}}^{\{0,k\}}$ -property, then X has $i\text{-}\overline{\text{DD}}^{\{0,j\}}$ -properties for all i, j with $i + j \leq m + k$:*

$$\mathcal{LR} \cap m\text{-}\overline{\text{DD}}^{\{0,k\}} \subset \bigcap_{i+j \leq m+k} i\text{-}\overline{\text{DD}}^{\{0,j\}}$$

- (3) *If either $X \in \overline{\mathcal{Z}}_{m+p}$ or $X \in \overline{\mathcal{Z}}_{m+p}^{\mathbb{Z}} \cap \text{LC}^1$, then the product $X \times Y$ has the $m\text{-}\overline{\text{DD}}^{\{n,k+p+1\}}$ -property for each separable metrizable LC^k -space Y possessing the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property with $n \leq k$. This can be written as*

$$(\mathcal{LR} \cap \text{LC}^1 \cap \overline{\mathcal{Z}}_{m+p}^{\mathbb{Z}}) \times (\text{LC}^k \cap m\text{-}\overline{\text{DD}}^{\{n,k\}}) \subset m\text{-}\overline{\text{DD}}^{\{n,k+p+1\}}$$

Remark 4. Since $\mathbb{R}^q \in \mathcal{Z}_{q-1}$ is rectifiable, Theorem 30(3) implies that the product $X \times \mathbb{R}^{m+p}$ has the $m\text{-}\overline{\text{DD}}^{\{n,k+p\}}$ -property for any separable metrizable LC^k -space having the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property with $n \leq k$. This result was established by W. Mitchell [WM, Theorem 4.3(1)] in the case X is a compact ANR and $m = 0$. Moreover, a particular case of Theorem 30(1) when X is an ANR and $m = 0$ was also established in [WM].

1.14. k -Root and Division Formulas for the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties. In this section we discuss k -root and division formulas for the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties, one of the most surprising features of these properties.

Theorem 31. (*k*-Root Formulas) *Let n be a non-negative integer or infinity and k be a positive integer.*

- (1) *If X is an LC^1 -space with $2\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property and $X^k \in (kn+k-1)\text{-}\overline{\text{DD}}^{\{0,0\}}$, then X has the $n\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property. This can be written as*

$$\boxed{\text{LC}^1 \cap 2\text{-}\overline{\text{DD}}^{\{0,0\}} \cap \sqrt[k]{(kn+k-1)\text{-}\overline{\text{DD}}^{\{0,0\}}} \subset n\text{-}\overline{\text{DD}}^{\{0,0\}}}$$

- (2) *If X is a separable metrizable LC^{kn+k-1} -space with the $0\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property and X^k has the $0\text{-}\overline{\text{DD}}^{\{0,kn+k-1\}}$ -property, then X has the $0\text{-}\overline{\text{DD}}^{\{0,n\}}$ -property. This can be written as*

$$\boxed{\text{LC}^{kn+k-1} \cap 0\text{-}\overline{\text{DD}}^{\{0,2\}} \cap \sqrt[k]{0\text{-}\overline{\text{DD}}^{\{0,kn+k-1\}}} \subset 0\text{-}\overline{\text{DD}}^{\{0,n\}}}$$

To write down division formulas for the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property, let us introduce two new classes in addition to the classes $\cup_G \mathcal{Z}_n^G$ and $\exists_o \cup_G \overline{\mathcal{Z}}_n^G$:

- $\mathcal{Z}_n^{\exists G}$ - the class of spaces X with all $x \in X$ being $\exists G$ -homological Z_n -points in X ;
- $\Delta \mathcal{Z}_n^{\exists G}$ - the class of spaces X whose diagonal Δ_X is an $\exists G$ -homological Z_n -set in X^2 .

Note that any at most n -dimensional polyhedron belongs to none of the last two classes.

Theorem 32. (Division Formulas) *Let $n \leq k$ be non-negative integers or infinity and m a non-negative integer.*

- (1) *An LC^1 -space with the $2\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property has the $n\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property provided $X \times Y$ has the $(n+m)\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property for some space Y whose diagonal Δ_Y fails to be a $\exists G$ -homological Z_m -set in Y^2 . This can be written as*

$$\boxed{\text{LC}^1 \cap 2\text{-}\overline{\text{DD}}^{\{0,0\}} \cap \frac{(n+m)\text{-}\overline{\text{DD}}^{\{0,0\}}}{\text{Top} \setminus \Delta \mathcal{Z}_m^{\exists G}} \subset n\text{-}\overline{\text{DD}}^{\{0,0\}}}$$

- (2) *A separable metrizable LC^{n+m} -space $X \in 0\text{-}\overline{\text{DD}}^{\{0,2\}}$ has the $0\text{-}\overline{\text{DD}}^{\{0,n\}}$ -property provided $X \times Y$ has the $0\text{-}\overline{\text{DD}}^{\{0,n+m\}}$ -property for some metrizable separable Baire LC^{n+m} -space Y that contains no non-empty open set $U \in \cup_G \overline{\mathcal{Z}}_m^G$.*

$$\boxed{\text{LC}^{n+m} \cap 0\text{-}\overline{\text{DD}}^{\{0,2\}} \cap \frac{0\text{-}\overline{\text{DD}}^{\{0,n+m\}}}{\text{Br} \cap \text{LC}^{n+m} \setminus \exists_o \cup_G \overline{\mathcal{Z}}_m^G} \subset 0\text{-}\overline{\text{DD}}^{\{0,n\}}}$$

- (3) *A separable metrizable LC^{k+m} -space $X \in \mathcal{Z}_{k+2}^{\mathbb{Z}}$ with the $0\text{-}\overline{\text{DD}}^{\{2,2\}}$ -property has the $0\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property provided $X \times Y$ has the $0\text{-}\overline{\text{DD}}^{\{n+m,k+m\}}$ -property for some metrizable separable LC^{k+m} -space $Y \notin \mathcal{Z}_m^{\exists G}$. This can be written as*

$$\boxed{\text{LC}^{k+m} \cap 0\text{-}\overline{\text{DD}}^{\{2,2\}} \cap \mathcal{Z}_{k+2}^{\mathbb{Z}} \cap \frac{0\text{-}\overline{\text{DD}}^{\{n+m,k+m\}}}{\text{LC}^{k+m} \setminus \mathcal{Z}_m^{\exists G}} \subset 0\text{-}\overline{\text{DD}}^{\{n,k\}}}$$

- (4) A separable metrizable LC^{k+m} -space $X \in \mathcal{Z}_{n+k+m}^{\mathbb{Z}}$ possessing the $0\text{-}\overline{\text{DD}}^{\{2,2\}}$ -property has the $0\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property provided $X \times Y$ has the $0\text{-}\overline{\text{DD}}^{\{n+m,n+m\}}$ -property for some metrizable separable LC^{n+m} -space $Y \notin \cup_G \mathcal{Z}_m^G$. This can be written as

$$\text{LC}^{k+m} \cap 0\text{-}\overline{\text{DD}}^{\{2,2\}} \cap \mathcal{Z}_{n+k+m}^{\mathbb{Z}} \cap \frac{0\text{-}\overline{\text{DD}}^{\{n+m,n+m\}}}{\text{LC}^{n+m} \setminus \cup_G \mathcal{Z}_m^G} \subset 0\text{-}\overline{\text{DD}}^{\{n,k\}}$$

1.15. Characterizing $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties with $m, n, k \in \{0, \infty\}$. In this subsection we apply the results obtained in preceding subsections to the case of $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties with $m, n, k \in \{0, \infty\}$. Let us note that the $0\text{-}\overline{\text{DD}}^{\{0,0\}}$ has been characterized in Proposition 8 while $\infty\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$ is equivalent to $0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$. So, it suffices to consider only the properties: $0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$, $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$, $\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}}$, and $0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$. These properties can be characterized in terms of homotopical or homological Z_∞ -points as follows:

Corollary 3. (1) A topological (LC^1) -space X has the $\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property iff all points of X are homotopical Z_∞ -points (iff all points of X are homotopical Z_2 -points and X has $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property):

$$\mathcal{Z}_2 \cap \infty\text{-}\overline{\text{DD}}^{\{0,0\}} \cap \text{LC}^1 \subset \infty\text{-}\overline{\text{DD}}^{\{0,\infty\}} = \mathcal{Z}_\infty$$

- (2) An LC^1 -space X has the $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property iff $X \in 2\text{-}\overline{\text{DD}}^{\{0,0\}}$ and all points of X are homological Z_∞ -points:

$$\mathcal{Z}_\infty^{\mathbb{Z}} \cap 2\text{-}\overline{\text{DD}}^{\{0,0\}} \cap \text{LC}^1 \subset \infty\text{-}\overline{\text{DD}}^{\{0,0\}} \subset \mathcal{Z}_\infty^{\mathbb{Z}}$$

- (3) A Polish LC^∞ -space X has the $0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property iff X has a dense set of homotopical Z_∞ -points (iff $X \in 0\text{-}\overline{\text{DD}}^{\{0,2\}}$ and X has a dense set of homological Z_∞ -points):

$$\overline{\mathcal{Z}}_\infty^{\mathbb{Z}} \cap 0\text{-}\overline{\text{DD}}^{\{0,2\}} \cap \text{LC}^\infty \subset 0\text{-}\overline{\text{DD}}^{\{0,\infty\}} \text{ and } 0\text{-}\overline{\text{DD}}^{\{0,\infty\}} \cap \text{LC}^\infty \cap \Pi_2^0 \subset \overline{\mathcal{Z}}_\infty$$

- (4) If each point of a metrizable separable LC^∞ -space X is a homological Z_∞ -point and X has the properties $\text{AP}[\infty]$ and $0\text{-}\overline{\text{DD}}^{\{2,2\}}$, then X has the $0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$ -property:

$$\mathcal{Z}_\infty^{\mathbb{Z}} \cap 0\text{-}\overline{\text{DD}}^{\{2,2\}} \cap \text{LC}^\infty \cap \text{AP}[\infty] \subset \infty\text{-}\overline{\text{DD}}^{\{\infty,\infty\}} = 0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$$

According to the famous characterization of Hilbert cube manifolds due to Toruńczyk [To2], a locally compact ANR-space X is a Q -manifold if and only if X has the $0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$ -property. Combining this characterization with the last item of Corollary 3 we obtain a new characterization of Q -manifolds.

Corollary 4. A locally compact ANR is a Q -manifold if and only if

- X has the disjoint disk property;
- each point of X is a homological Z_∞ -point;
- each map $f : \mathbb{I}^\infty \rightarrow X$ can be uniformly approximated by maps with trt -dimensional image.

Next, we discuss k -root formulas for $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties with $m, n, k \in \{0, \infty\}$.

Corollary 5. (k -Root Formulas)

- (1) An LC^1 -space X has the $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property if and only if $X \in 2\text{-}\overline{\text{DD}}^{\{0,0\}}$ and $X^k \in \infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ for some finite k . This can be written as

$$\infty\text{-}\overline{\text{DD}}^{\{0,0\}} \supset \sqrt[k]{\infty\text{-}\overline{\text{DD}}^{\{0,0\}}} \cap 2\text{-}\overline{\text{DD}}^{\{0,0\}} \cap \text{LC}^1$$

- (2) A metrizable separable LC^∞ -space X has the $0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property if and only if $X \in 0\text{-}\overline{\text{DD}}^{\{0,2\}}$ and $X^k \in 0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ for some finite k . This can be written as

$$0\text{-}\overline{\text{DD}}^{\{0,\infty\}} \supset \sqrt[k]{0\text{-}\overline{\text{DD}}^{\{0,\infty\}}} \cap 0\text{-}\overline{\text{DD}}^{\{0,2\}} \cap \text{LC}^\infty$$

- (3) An LC^1 -space X has the $\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property iff X has the $2\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property and $X^k \in \infty\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ for some finite k . This can be written as

$$\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}} \supset \sqrt[k]{\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}}} \cap 2\text{-}\overline{\text{DD}}^{\{0,2\}} \cap \text{LC}^1$$

- (4) A metrizable separable LC^∞ -space X has the $0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$ -property if X has the $0\text{-}\overline{\text{DD}}^{\{2,2\}}$ -property, $X \in \text{AP}[\infty]$ and $X^k \in 0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$ for some finite k . This can be written as

$$0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}} \supset \sqrt[k]{0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}} \cap 0\text{-}\overline{\text{DD}}^{\{2,2\}} \cap \text{LC}^\infty \cap \text{AP}[\infty]$$

Finally, we turn to division formulas for the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties with $m, n, k \in \{0, \infty\}$.

Corollary 6. (Division Formulas)

- (1) An LC^1 -space X has the $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property provided X has the $2\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property and the product $X \times Y$ has the $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property for some space $Y \notin \cup_G \mathcal{Z}_\infty^G$. This can be written as

$$\infty\text{-}\overline{\text{DD}}^{\{0,0\}} \supset \frac{\infty\text{-}\overline{\text{DD}}^{\{0,0\}}}{\text{Top} \setminus \cup_G \mathcal{Z}_\infty^G} \cap 2\text{-}\overline{\text{DD}}^{\{0,0\}} \cap \text{LC}^1$$

- (2) An LC^1 -space X has the $\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property provided X has the $2\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property and the product $X \times Y$ has the $\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property for some space $Y \notin \cup_G \mathcal{Z}_\infty^G$. This can be written as

$$\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}} \supset \frac{\infty\text{-}\overline{\text{DD}}^{\{0,\infty\}}}{\text{Top} \setminus \cup_G \mathcal{Z}_\infty^G} \cap 2\text{-}\overline{\text{DD}}^{\{0,2\}} \cap \text{LC}^1$$

- (3) A metrizable separable LC^∞ -space X has the $0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$ -property provided $X \in 0\text{-}\overline{\text{DD}}^{\{2,2\}}$ and $X \times Y$ has the $0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$ -property for some separable metrizable LC^∞ -space $Y \notin \cup_G \mathcal{Z}_\infty^G$. This can be written as

$$0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}} \supset \frac{\text{LC}^\infty \cap 0\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}}{\text{Top} \setminus \cup_G \mathcal{Z}_\infty^G} \cap 0\text{-}\overline{\text{DD}}^{\{2,2\}}$$

- (4) A metrizable separable LC^∞ -space X has the $0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property provided $X \in 0\text{-}\overline{\text{DD}}^{\{0,2\}}$ and the product $X \times Y$ has the $0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$ -property for some metrizable separable LC^∞ -space $Y \notin \exists_o \cup_G \mathcal{Z}_\infty^G$. This can be written as

$$0\text{-}\overline{\text{DD}}^{\{0,\infty\}} \supset \frac{\text{LC}^\infty \cap 0\text{-}\overline{\text{DD}}^{\{0,\infty\}}}{\text{Top} \setminus \exists_o \cup_G \overline{\mathcal{Z}}_\infty^G} \cap 0\text{-}\overline{\text{DD}}^{\{0,2\}}$$

The third item of Corollary 6 combined with the Toruńczyk's characterization of Q -manifolds implies the following division theorem for Q -manifolds proven in [BC] and implicitly in [DW].

Corollary 7. *A space X is a Q -manifold if and only if the product $X \times Y$ is a Q -manifold for some space $Y \notin \cup_G \mathcal{Z}_\infty^G$.*

1.16. Dimension of spaces with the $m\text{-}\overline{\text{DD}}^n$ -property. In this section we study the dimensional properties of spaces possessing the $m\text{-}\overline{\text{DD}}^n$ -property.

Theorem 33. *If a metrizable separable space X has the $m\text{-}\overline{\text{DD}}^n$ -property, then $\dim X \geq n + \frac{m+1}{2}$.*

This theorem combined with Theorem 13 allows us to calculate the smallest possible dimension of a space X with $m\text{-}\overline{\text{DD}}^n$. For a real number r let

$$\lfloor r \rfloor = \max\{n \in \mathbb{Z} : n \leq r\}$$

$$\lceil r \rceil = \min\{n \in \mathbb{Z} : n \geq r\}.$$

Corollary 8. *Let n, m be non-negative integers and D be a dendrite with a dense set of end-points.*

- (1) *If m is odd and $d = n + \frac{m+1}{2}$, then the power D^d is a d -dimensional absolute retract with the $m\text{-}\overline{\text{DD}}^n$ -property.*
- (2) *If m is even and $d = n + \frac{m+2}{2}$, then the product $D^{d-1} \times \mathbb{I}$ is a d -dimensional absolute retract with the $m\text{-}\overline{\text{DD}}^n$ -property.*

Consequently, $n + \lceil \frac{m+1}{2} \rceil$ is the smallest possible dimension of a compact absolute retract with the $m\text{-}\overline{\text{DD}}^n$ -property.

Theorem 33 implies that $\dim(X) \geq m+1$ for each metrizable separable space X with the $(2m+1)\text{-}\overline{\text{DD}}^0$ -property. A similar inequality also holds also for cohomological dimension.

Theorem 34. *Let X be a locally compact metrizable LC^m -space having the $(2m+1)\text{-}\overline{\text{DD}}^0$ -property. Then $\dim_G X \geq m+1$ for any non-trivial Abelian group G .*

In some cases the condition $X \in (2m+1)\text{-}\overline{\text{DD}}^0$ from Theorem 34 can be weakened to $X \in 2m\text{-}\overline{\text{DD}}^0$.

Theorem 35. *Let X be a locally compact LC^{2m} -space with the $2m\text{-}\overline{\text{DD}}^0$ -property and let G be a non-trivial Abelian group. The inequality $\dim_G X \geq m+1$ holds in each of the following cases:*

- (1) *G fails to be both divisible and periodic;*
- (2) *G is a field;*
- (3) *X is an ANR-space.*

Theorem 35 implies the following estimation for the extension dimension of spaces $X \in m\text{-}\overline{\text{DD}}^{\{0,0\}}$:

Theorem 36. *Let X be a locally compact LC^m -space such that $\text{e-dim} X \leq L$ for some CW-complex L . If $X \in m\text{-}\overline{\text{DD}}^0$, then we have:*

- (1) *The homotopy groups $\pi_i(L)$ are trivial for all $i < \frac{m}{2}$;*
- (2) *For $n = \lfloor m/2 \rfloor$ the group $\pi_n(L)$ is both divisible and periodic, and $\pi_n(L) = \tilde{H}_n(L)$;*
- (3) *$\pi_i(L) = 0$ for all $i \leq \frac{m}{2}$ provided X is an ANR-space.*

Finally, we discuss the dimension properties of spaces $X \in \infty\text{-}\overline{\text{DD}}^{\{0,0\}}$.

Theorem 37. *Let X be a locally compact metrizable LC^∞ -space with the $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property. Then*

- (1) *All points of X are homological Z_∞ -points;*
- (2) *X fails to be trt-dimensional;*
- (3) *If $\text{e-dim} X \leq L$ for some CW-complex L , then L is contractible;*
- (4) *If X is locally contractible, then X is not a C -space.*

The first item of this theorem follows from Theorem 10(7). The last three items follow from Theorem 23(1) and Theorem 24.

1.17. Some Examples and Open Problems. First, we discuss the problem of distinguishing between the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties for various m, n, k . Let us note that if an Euclidean space E has the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property for some m, n, k , then E has the $a\text{-}\overline{\text{DD}}^{\{b,c\}}$ -property for all non-negative integers a, b, c with $a + b + c \leq n + m + k$. This feature is specific for Euclidean spaces and does not hold in the general case. For example, each dendrite D with a dense set of end-points has the $0\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property (and in fact, $0\text{-}\overline{\text{DD}}^{\{0,\infty\}}$) but doesn't have the $0\text{-}\overline{\text{DD}}^{\{1,1\}}$ -property. Next example from Daverman's book [Da2] shows that the properties $0\text{-}\overline{\text{DD}}^{\{0,2\}}$ and $0\text{-}\overline{\text{DD}}^{\{1,1\}}$ are completely incomparable.

Example 1. There is a 2-dimensional absolute retract $\Lambda \subset \mathbb{R}^3$ with $0\text{-}\overline{\text{DD}}^{\{1,1\}}$ -property that fails to have the $0\text{-}\overline{\text{DD}}^{\{0,2\}}$ -property.

Question 1. Does the space Λ from Example 1 possess the $2\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property?

It follows from Theorem 32(1) that a Polish LC^∞ -space X has the $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property provided $X \times \mathbb{R} \in \infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ and $X \in 2\text{-}\overline{\text{DD}}^{\{0,0\}}$. We do not know if the latter condition is essential.

Question 2. Does a compact absolute retract X possess the $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property provided $X \times \mathbb{I}$ has that property? (Let us observe that $X \times \mathbb{I} \in \infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ implies $X \times \mathbb{I} \in \infty\text{-}\overline{\text{DD}}^{\{1,\infty\}}$).

This question is equivalent to another intriguing one

Question 3. Does a compact absolute retract X contain a Z_2 -point provided all points of the product $X \times \mathbb{I}$ are Z_∞ -points?

Problem 3. *Let X be a compact AR with the $\infty\text{-}\overline{\text{DD}}^{\{0,0\}}$ -property.*

- (1) *Is there any Z_2 -point in X ?*
- (2) *Is X strongly infinite-dimensional?*
- (3) *Is $X \times \mathbb{I}$ homeomorphic to the Hilbert cube?*

Problem 4. *Is a space $X \in 0\text{-}\overline{\text{DD}}^{\{2,2\}}$ homeomorphic to the Hilbert cube Q provided some finite power of X is homeomorphic to Q ?*

There are three interesting examples relevant to these questions. The first of them was constructed by Singh in [Sin], the second by Daverman and Walsh in [DW] and the third by Banach and Repovš in [BR].

Example 2 (Singh). There is a space X possessing the following properties:

- (1) X is a compact absolute retract;
- (2) X contains no topological copy of the 2-disk \mathbb{I}^2 ;
- (3) $X \times \mathbb{I}$ is homeomorphic to the Hilbert cube;
- (4) All points of X except for countably many are Z_2 -points;
- (5) $X \in \infty\text{-}\overline{\text{DD}}^{\{0,0\}}$;
- (6) $X \notin 2\text{-}\overline{\text{DD}}^{\{0,2\}} \cup 0\text{-}\overline{\text{DD}}^{\{2,2\}}$;
- (7) $X \times \mathbb{I} \in \infty\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$.

Example 3 (Daverman-Walsh). There is a space X possessing the following properties:

- (1) X is a compact absolute retract;
- (2) $X \times \mathbb{I}$ is homeomorphic to the Hilbert cube;
- (3) each point of X is a Z_∞ -point;
- (4) $X \in \infty\text{-}\overline{\text{DD}}^{\{0,\infty\}} \cap 0\text{-}\overline{\text{DD}}^{\{1,\infty\}}$;
- (5) $X \notin 0\text{-}\overline{\text{DD}}^{\{2,2\}}$;
- (6) $X \times \mathbb{I} \in \infty\text{-}\overline{\text{DD}}^{\{\infty,\infty\}}$.

Example 4 (Banach-Repovš). There is a countable family \mathcal{X} of spaces such that

- (1) the product $X \times Y$ of any two different spaces $X, Y \in \mathcal{X}$ is homeomorphic to the Hilbert cube;
- (2) no finite power X^k of any space $X \in \mathcal{X}$ is homeomorphic to Q .

It is interesting to note that there is no uncountable family \mathcal{X} possessing the properties (1) and (2) from Example 4, see [BR].

It may be convenient to describe the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -properties of a space X using the following sets

$$\begin{aligned} * \text{-}\overline{\text{DD}}^{\{*,* \}}(X) &= \{(m, n, k) \in \omega^3 : X \text{ has the } m\text{-}\overline{\text{DD}}^{\{n,k\}}\text{-property}\}, \\ 0\text{-}\overline{\text{DD}}^{\{*,* \}}(X) &= \{(n, k) \in \omega^2 : X \text{ has the } 0\text{-}\overline{\text{DD}}^{\{n,k\}}\text{-property}\}, \\ * \text{-}\overline{\text{DD}}^*(X) &= \{(m, n) \in \omega^2 : X \text{ has the } m\text{-}\overline{\text{DD}}^n\text{-property}\}. \end{aligned}$$

Problem 5. *Describe the geometry of the sets $* \text{-}\overline{\text{DD}}^{\{*,* \}}(X)$, $0\text{-}\overline{\text{DD}}^{\{*,* \}}(X)$ and $* \text{-}\overline{\text{DD}}^*(X)$ for a given space X . Which subsets of ω^3 or ω^2 can be realized as the sets $* \text{-}\overline{\text{DD}}^{\{*,* \}}(X)$, $0\text{-}\overline{\text{DD}}^{\{*,* \}}(X)$ or $* \text{-}\overline{\text{DD}}^*(X)$ for a suitable X ?*

In fact, we can consider the following partial pre-order $\xRightarrow[\text{DD}]{}_{\text{DD}}$ on the set ω^3 :

$(m, n, k) \xRightarrow[\text{DD}]{}_{\text{DD}} (a, b, c)$ if each space X with the $m\text{-}\overline{\text{DD}}^{\{n,k\}}$ -property has also the $a\text{-}\overline{\text{DD}}^{\{b,c\}}$ -property.

Problem 6. *Describe the properties of the partial preorder $\xRightarrow[\text{DD}]{}_{\text{DD}}$ on ω^3 .*

By Proposition 3(2), a paracompact space X is Lefschetz ANE $[n]$ for a finite n if and only if X is an LC^{n-1} -space. Consequently, the product of two paracompact ANE $[n]$ -spaces is an ANE $[n]$ -space for every finite n .

Problem 7. *Is the product of two (paracompact) Lefschetz $\text{ANE}[\infty]$ -spaces a Lefschetz $\text{ANE}[\infty]$ -space?*

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